CONVEXITY IN A HADLEY-WHITIN MODEL.

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THESIS

CONVEXITY IN A HADLEY-WHITIN MODEL

by

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I. INTRODUCTION

In any inventory model, one of the main objectives is to avoid both overstock and out-of-stock situations. The latter is especially critical in military supply. In lot size - reorder point models with stochastic demands, a decision has to be made as to when to reorder and how much the reorder quantity should be. The goal, then, is to achieve in some sense an optimal reorder level r and an optimal reorder quantity Q. One way to accomplish this goal is to define a cost function K which then acts as an objective function to be minimized by choice of Q and r as variables.

Hadley and Whitin [Ref. 1] present several lot size - reorder point models. The one of interest in this paper is called a backorder model wherein, if demands occur when the system is out of stock, they will be backordered with some penalty cost, to be called the backorder cost. Customers then must wait until all orders are delivered to the inventory manager. Associated with delivery is a delay in time called lead time. In this paper we consider only the case where lead-time demand is assumed to be normally distributed.

Mathematically, the backorder model of interest here has the objective function,

$$K(Q,r) = \frac{\lambda A}{Q} + IC[\frac{Q}{2} + r - \mu] + \frac{\pi\lambda}{Q}[(\mu-r)\Phi(\frac{r-\mu}{\sigma}) + \sigma\phi(\frac{r-\mu}{\sigma})] \qquad (1)$$

where

A is the set-up cost (cost of placing an order)

π is the backorder cost per unit backordered

λ is the average annual demand

I is the inventory carrying charge

C is the unit cost of the item

μ and σ are the mean and standard deviation of the normally distributed lead-time demand random variable

$\phi(z)$ is the standard normal density function

$\Phi(z)$ is its complementary cumulative distribution function, i.e., $\Phi(z) = \int_z^\infty \phi(x)\,dx$.

$K(Q,r)$ is called the <u>average annual cost</u> and is composed of three different types of cost: the ordering cost, the cost of carrying inventory and the cost of backorders. These costs are represented, respectively, by the first, second and third term in the expression for $K(Q,r)$ in (1).

For ease in the algebra, (1) will be rewritten as

$$K(Q,z) = \frac{\lambda A}{Q} + IC[\frac{Q}{2} + \sigma z] + \frac{\pi\lambda\sigma}{Q}[\phi(z) - z\Phi(z)] \tag{2}$$

where $z = \frac{r-\mu}{\sigma}$.

Hadley and Whitin [Ref. 1, pp. 165 ff.] point out that the model just defined is appropriate when the expected number of backorders is negligible. They then propose a solution to the optimization problem by claiming that K is convex in Q

and r (or z) and therefore any solution obtained by setting the partial derivatives of K equal to zero will determine a global minimum. They provide an iterative scheme for such a solution along with the assertion that the solution, when it exists, is unique. Finally, they discuss the fact that a solution will not always exist, pointing out that this "anomaly" arises because a backorders term was omitted in the expression for the carrying cost. Since the model is to apply only when such a term is negligible, they assert that this situation will never occur when high backorder costs are involved without being explicit as to what "high" means.

To verify these results, the reader is directed to solve a series of exercises, the first of which is to show that the function involving the backorder cost (the third term) in (1), say $J(Q,r)$, is convex in Q and r. It would then follow immediately that K is convex in Q and r. But it was pointed out as early as 1964 by Veinott [Ref. 2] that J is not convex. In 1969, Brooks and Lu [Ref. 3] addressed the same problem and derived a general result which, when applied to the normal case, shows that J is convex for all $r \geq \mu$, the mean lead-time demand.

However, these results still leave the main issue unresolved. In the first place, solutions can be obtained where $r < \mu$.[1] More importantly, the real issue is whether

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[1]The difference $r-\mu$ is sometimes referred to as a <u>safety level</u>. In that case, a negative safety level simply means it is more economical to tolerate a backorder position on the average than to protect against that position.

or not K is convex, and, in any case, how its minimum value is characterized. The purpose of this thesis is to clarify this issue and characterize the solutions to the problem in a more explicit manner.

Section II will show that in fact K is not convex and, in addition, will exhibit an example in which the first partial derivatives $\frac{\partial K}{\partial Q}$ and $\frac{\partial K}{\partial z}$ vanish at two different points. Thus, the solution is not unique as claimed by Hadley and Whitin. Section III will deal with the characterization of the solutions and will show that under suitable conditions there are always two distinct solutions, one of them being a minimum and the other a saddle point. Conditions for the existence of a solution will be given in terms of the set-up cost A. Section IV will discuss the application of the results in Section III to practical inventory problems along with the iterative scheme given by Hadley and Whitin to solve for a solution.

II. COUNTEREXAMPLES

As pointed out in the introduction, the backorders term in the cost function is not convex as asserted by Hadley and Whitin [Ref. 1, Problem 4-6, p. 221]. Even so, it might still be the case that K is convex as asserted by the authors. That this is not so will be demonstrated in part A. On the other hand, even though K is not convex, it might still be the case that the partial derivatives vanish at a unique point and that point determines a minimum. That this is not so will be shown in part B and a characterization of solutions will be given in the next section.

A. CONVEXITY CONDITION FOR K

The function K is convex if and only if its Hessian matrix Ω is positive semi-definite [Ref. 4]. Since K is a function of two variables, the matrix Ω will be positive semi-definite when its diagonal terms and its determinant are nonnegative.

For the case at hand, using (2), the Hessian matrix for K is given by

$$\Omega = \begin{bmatrix} \frac{2\lambda}{Q^3}\left(A + \pi\bar{\eta}(z)\right) & \frac{\pi\lambda\Phi(z)}{Q^2} \\ \\ \frac{\pi\lambda\Phi(z)}{Q^2} & \frac{\pi\lambda\phi(z)}{\sigma Q} \end{bmatrix}$$

where $\bar{\eta}(z) = \sigma\phi(z) - \sigma z\Phi(z)$

or $\bar{\eta}(z) = \sigma u(z)$ (3)

where $u(z) = \phi(z) - z\Phi(z)$. (4)

In Appendix A it is shown that $u(z) > 0$ for all z . Hence, for nonnegative set-up costs A, it is clear that the diagonal terms of Ω are positive. Thus K is convex if and only if $\det \Omega \geq 0$.

Now,

$$\det \Omega = \frac{2\pi\lambda^2\phi(z)}{\sigma Q^4}(A + \pi\bar{\eta}(z)) - \frac{\pi\lambda^2\Phi^2(z)}{Q^4}$$

and thus $\det \Omega \geq 0$ if and only if

$$\frac{2\pi\lambda^2\phi(z)}{\sigma Q^4}(A + \pi\bar{\eta}(z)) - \frac{\pi^2\lambda^2\Phi^2(z)}{Q^4} \geq 0$$

Multiplying the above inequality by $Q^4 > 0$ and dividing by $\pi\lambda^2 > 0$ gives

$$\frac{2\phi(z)}{\sigma}(A + \pi\bar{\eta}(z)) - \pi\Phi^2(z) \geq 0$$

or $$\frac{2A}{\sigma}\phi(z) + 2\pi\phi^2(z) - 2\pi z\phi(z)\Phi(z) - \pi\Phi^2(z) \geq 0 \qquad (5)$$

Defining V(z) to be equal to the left-hand side of (5), then the following summarizes convexity.

RESULT: The function K is convex if and only if $\det \Omega \geq 0$ or equivalently if and only if $V(z) \geq 0$.

REMARK: This result holds for any $Q > 0$ and it is shown in Appendix A that $V(z) > 0$ whenever $z \geq 0$.

A counterexample to the convexity of K is then attainable from an example provided by Hadley and Whitin [Ref. 1, p. 173]. There the parameters chosen are $\pi = 2000$, $\lambda = 1600$, $C = 50$, $I = .20$, $A = 4000$, $\mu = 750$, $\sigma = 50$. For such a case $V(z)$ becomes

$$V(z) = 160\phi(z) + 4000\phi^2(z) - 4000z\phi(z)\Phi(z) - 2000\Phi^2(z)$$

and for $z = -1$ (equivalently, $r = 700$)

$$V(-1) \doteq 38.72 + 234.256 + 814.3784 - 1415.57138$$

$$= -328.21698 < 0$$

Thus, K cannot be convex.

B. COUNTEREXAMPLE TO UNIQUENESS OF THE SOLUTION

DEFINITION. (Q,z) will be called a solution whenever $Q > 0$ and z satisfy the following equations:

$$0 = \frac{\partial K}{\partial Q} = -\frac{\lambda A}{Q^2} + \frac{IC}{2} - \frac{\pi\lambda\bar{\eta}(z)}{Q^2} = -\frac{\lambda A}{Q^2} + \frac{IC}{2} - \frac{\pi\lambda\sigma u(z)}{Q^2} \qquad (6)$$

$$0 = \frac{\partial K}{\partial z} = (IC - \frac{\pi\lambda\Phi(z)}{Q})\sigma \qquad (7)$$

It is convenient to write (6) and (7) as:

$$Q = \sqrt{\frac{2\lambda(A + \pi\bar{\eta}(z))}{IC}} = \sqrt{\frac{2\lambda(A + \pi\sigma u(z))}{IC}} \tag{8}$$

and

$$\Phi(z) = \frac{QIC}{\pi\lambda} \tag{9}$$

COUNTEREXAMPLE. Let $\pi = \lambda = \sigma = 1$, $I = .1$, $C = 2.42$ and $\mu = 3$. Let $A = \frac{1}{2(.1)(2.42)} \Phi^2(.05) - u(.05) \doteq .1018$, where $u(z)$ is defined in (4). Then there are two distinct solutions

1) $(Q_1, z_1) = \left(\frac{\Phi(z_1)}{(.1)(2.42)}, z_1 \right)$ where $z_1 = .05$

2) $(Q_2, z_2) = \left(\frac{\Phi(z_2)}{(.1)(2.42)}, z_2 \right)$ where $-1.81 < z_2 < -1.80$

To see that (Q_1, z_1) is a solution,

$$\left.\frac{\partial K}{\partial Q}\right|_{Q_1, z_1} = - \frac{1 \left[\frac{\Phi^2(.05)}{2(.1)(2.42)} - u(.05)\right]}{\frac{\Phi^2(.05)}{(.1)^2(2.42)^2}} + \frac{(.1)(2.42)}{2} - \frac{(1)(1)(1)u(.05)}{\frac{\Phi^2(.05)}{(.1)^2(2.42)^2}}$$

$$= - \frac{(.1)(2.42)}{2} + \frac{(.1)^2(2.42)^2 u(.05)}{\Phi^2(.05)} + \frac{(.1)(2.42)}{2} - \frac{(.1)^2(2.42)^2 u(.05)}{\Phi^2(.05)}$$

$= 0$. Thus, equation (6) is satisfied.

$$\left.\frac{\partial K}{\partial z}\right|_{Q_1, z_1} = (.1)(2.42) - \frac{(1)(1)\Phi(.05)}{\frac{\Phi(.05)}{(.1)(2.42)}} = (.1)(2.42) - (.1)(2.42) = 0 \cdot$$

Hence equation (7) is satisfied. Therefore, (Q_1, z_1) is a solution.

Now, let $S(z) = \frac{1}{2(.1)(2.42)} \Phi^2(z) - u(z)$. Since S is clearly a continuous function and

$$S(-1.81) \leq .0997 < A < .1060 \leq S(-1.80)$$

there must exist some z_2 such that $-1.81 < z_2 < -1.80$ and $S(z_2) = A$. The verification for (Q_2, z_2) is done in a similar manner.

$$\left.\frac{\partial K}{\partial Q}\right|_{Q_2, z_2} = -\frac{1\left[\frac{\Phi^2(z_2)}{2(.1)(2.42)} - u(z_2)\right]}{\frac{\Phi^2(z_2)}{(.1)^2(2.42)^2}} + \frac{(.1)(2.42)}{2} - \frac{(1)(1)(1)u(z_2)}{\frac{\Phi^2(z_2)}{(.1)^2(2.42)^2}}$$

$$= -\frac{(.1)(2.42)}{2} + \frac{(.1)^2(2.42)^2 u(z_2)}{\Phi^2(z_2)} + \frac{(.1)(2.42)}{2} - \frac{(.1)^2(2.42)^2 u(z_2)}{\Phi^2(z_2)}$$

$= 0$. Thus, equation (6) is satisfied.

$$\left.\frac{\partial K}{\partial z}\right|_{Q_2, z_2} = (.1)(2.42) - \frac{(1)(1)\Phi(z_2)}{\frac{\Phi(z_2)}{(.1)(2.42)}} = (.1)(2.42) - (.1)(2.42) = 0$$

and hence equation (7) is satisfied. Therefore, (Q_2, z_2) is also a solution.

To illustrate these results, a graph similar to the one given on p. 171 of Ref. 1 is reproduced in Fig. 2-1 for this special case. In this graph $r = \sigma z + \mu$ is expressed as a function of Q for each of Equations (8) and (9). It shows that the two curves intersect at more than one point.

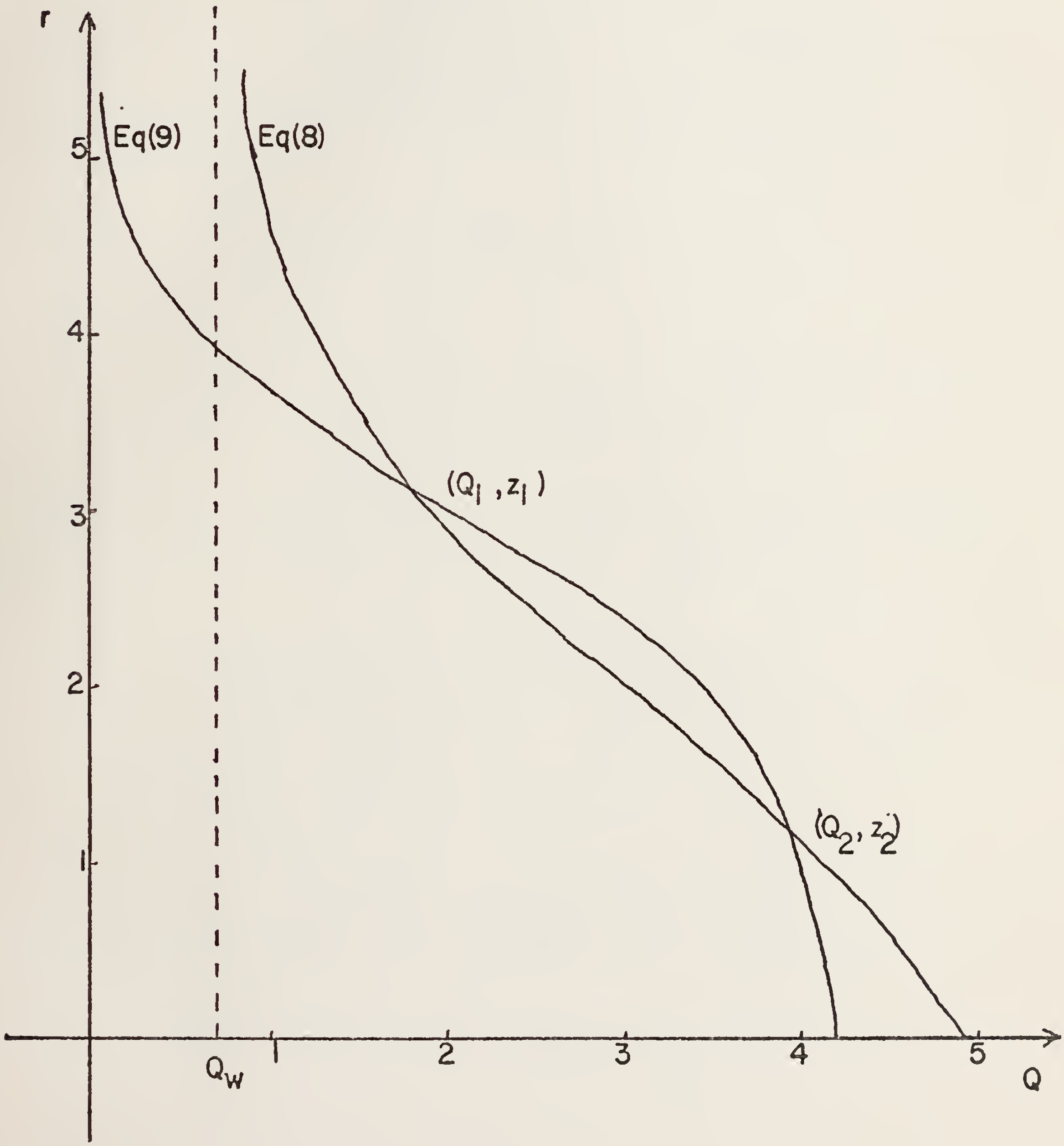

Fig. 2-1.

The value $A = \frac{1}{2(.1)(2.42)} \Phi^2(.05) - u(.05)$ was by no means chosen by accident. As it will be seen in the next section, the set-up cost A plays an important role in the characterization of the solutions, and it will be shown that, under appropriate conditions of the other parameters, there are always two distinct solutions, one of which is a minimum.

III. CHARACTERIZATION OF MINIMUM SOLUTION

Let a set of parameters be given. To simplify matters, define

$$S(z) = \frac{\pi^2\lambda}{2IC}\,\phi^2(z) - \pi\sigma u(z) \tag{10}$$

A series of lemmas are convenient to derive the main result in a theorem.

LEMMA 1. Let (Q^*,z^*) be a solution. Then,

(a) $A = S(z^*)$

(b) $V(z^*) \geq 0$ if and only if $\phi(z^*) \geq \frac{IC\sigma}{\pi\lambda}$ and $V(z^*) = 0$ if and only if $\phi(z^*) = \frac{IC\sigma}{\pi\lambda}$.

Proof (a). By definition of a solution, Q^* and z^* must satisfy (8) and (9) of Section III. Thus,

$$\frac{2\pi(A+\pi\sigma u(z^*))}{IC} = (Q^*)^2 = \frac{(\pi\lambda)^2\phi^2(z^*)}{(IC)^2}$$

or

$$2\lambda A + 2\pi\lambda\sigma u(z^*) = \frac{(\pi\lambda)^2}{IC}\,\phi^2(z^*)$$

$$2\lambda A = \frac{(\pi\lambda)^2}{IC}\,\phi^2(z^*) - 2\pi\lambda\sigma u(z^*)$$

$$A = \frac{\pi^2\lambda}{2IC}\,\phi^2(z^*) - \pi\sigma u(z^*) = S(z^*)$$

Proof (b).

$$V(z^*) = \frac{2A}{\sigma}\,\phi(z^*) + 2\pi\phi(z^*)(\phi(z^*)-z^*\Phi(z^*)) - \pi\Phi^2(z^*)$$

$$= \frac{2A}{\sigma}\,\phi(z^*) + 2\pi\phi(z^*)u(z^*) - \pi\Phi^2(z^*)$$

$$= 2\phi(z^*)(\frac{A}{\sigma} + \pi u(z^*)) - \pi\Phi^2(z^*)$$

$$= 2\phi(z^*)(\frac{\pi^2\lambda}{2IC\sigma}\Phi^2(z^*)-\pi u(z^*)+\pi u(z^*)) - \pi\Phi^2(z^*) \text{ from part (a).}$$

$$= \frac{\pi^2\lambda}{IC\sigma}\,\phi(z^*)\Phi^2(z^*) - \pi\Phi^2(z^*)$$

$$= \frac{\pi^2\lambda}{IC\sigma}\,\Phi^2(z^*)\;[\phi(z^*) - \frac{IC\sigma}{\pi\lambda}]$$

Since $\frac{\pi^2\lambda}{IC\sigma}\,\Phi^2(z^*) > 0$, the result stated in (b) follows immediately.

Q.E.D.

It is not always true that there exists a solution for any given set of parameters. The following lemma discusses conditions for which K cannot be minimized.

LEMMA 2. Given $A \geq 0$ and π, λ, I, C, σ all positive, a necessary condition for the function K to be minimized is that $\frac{IC\sigma}{\pi\lambda} < \phi(0)$.

Proof. The proof is divided into two cases.

Case (i): Suppose $\frac{IC\sigma}{\pi\lambda} > \phi(0)$ and K is minimized at (Q^*,z^*). Then the partial derivatives must vanish at (Q^*,z^*). But

$$V(z^*) = \frac{\pi^2\lambda}{IC\sigma} \phi^2(z^*)[\phi(z^*) - \frac{IC\sigma}{\pi\lambda}] < 0$$

since $\phi(z^*) \leq \phi(0)$ and so (Q^*,z^*) is a saddle point. This is a contradiction.

<u>Case (ii)</u>: Suppose $\frac{IC\sigma}{\pi\lambda} = \phi(0)$ and K is minimized at (Q^*,z^*). Again, the partial derivatives must vanish at (Q^*,z^*). Now if $z^* \neq 0$ then $V(z^*) < 0$ and hence (Q^*,z^*) is a saddle point. Thus it must be the case that $z^* = 0$. But then, by Lemma 1, part (a),

$$A = \frac{\pi^2\lambda}{2IC} \phi^2(0) - \pi\sigma u(0)$$

$$= \frac{\pi^2\lambda}{8IC} - \pi\sigma\phi(0)$$

$$= \frac{\pi\sigma}{8} \cdot \frac{\pi\lambda}{IC\sigma} - \pi\sigma\phi(0)$$

$$= \frac{\pi\sigma}{8} \cdot \frac{1}{\phi(0)} - \pi\sigma\phi(0)$$

$$= \pi\sigma(\frac{1 - 8\phi^2(0)}{\phi(0)})$$

Since $1 - 8\phi^2(0) < 0$, $A < 0$ contradicting the fact that $A \geq 0$.

Thus K cannot be minimized unless $\frac{IC\sigma}{\pi\lambda} < \phi(0)$.

Q.E.D.

LEMMA 3. Suppose $\frac{IC\sigma}{\pi\lambda} < \phi(0)$. Let $z_o > 0$ and $-z_o < 0$ be the unique points such that $\phi(-z_o) = \phi(z_o) = \frac{IC\sigma}{\pi\lambda}$.

Then, with S(z) defined in (10),

(a) S has a minimum at z_o where $A_o' = S(z_o) < 0$

(b) S has a maximum at $-z_o$ where $A_o = S(-z_o) \geq 0$

(c) S is strictly concave increasing on the interval $(-\infty,-z_o)$.

Proof. $S(z) = \frac{\pi^2\lambda}{2IC}\Phi^2(z) - \pi\sigma u(z)$

$$S'(z) = -\frac{\pi^2\lambda}{IC}\phi(z)\Phi(z) - \pi\sigma u'(z)$$

$$= -\frac{\pi^2\lambda}{IC}\phi(z)\Phi(z) - \pi\sigma(-\Phi(z)) \qquad \text{(see Appendix A)}$$

$$= \pi\sigma\Phi(z) - \frac{\pi^2\lambda}{IC}\phi(z)\Phi(z)$$

$$= \frac{\pi^2\lambda}{IC}\Phi(z)\left[\frac{IC\sigma}{\pi\lambda} - \phi(z)\right]$$

Since $\frac{\pi^2\lambda}{IC}\Phi(z) > 0$, the first derivative S'(z) vanishes only at $-z_o$ and z_o. Also, $S'(z) < 0$ for $-z_o < z < z_o$ and $S'(z) > 0$ for $z < -z_o$ or $z > z_o$; that is, S is decreasing in the interval $(-z_o,z_o)$ and increasing in the intervals $(-\infty,-z_o)$ and (z_o,∞).

$$S''(z) = -\frac{\pi^2\lambda}{IC}\phi(z)\left(\frac{IC\sigma}{\pi\lambda} - \phi(z)\right) + \frac{\pi^2\lambda}{IC}\Phi(z)(z\phi(z))$$

$$= -\pi\sigma\phi(z) + \frac{\pi^2\lambda\phi^2(z)}{IC} + \frac{\pi^2\lambda z\phi(z)\Phi(z)}{IC}$$

$$= \frac{\pi^2\lambda\phi(z)}{IC}(\phi(z) + z\Phi(z)) - \pi\sigma\phi(z)$$

$$= \pi\sigma\phi(z)\left[\frac{\pi\lambda}{IC\sigma}(\phi(z) + z\Phi(z)) - 1\right]$$

Thus, at z_o,

$$S''(z_o) = \pi\sigma\phi(z_o)\left[\frac{\pi\lambda}{IC\sigma}\left(\frac{IC\sigma}{\pi\lambda} + z_o\Phi(z_o)\right) - 1\right]$$

$$= \pi\sigma\phi(z_o)\left[1 + \frac{\pi\lambda}{IC\sigma} z_o\Phi(z_o) - 1\right]$$

$$= \frac{\pi^2\lambda}{IC} z_o\phi(z_o)\Phi(z_o) > 0 \qquad (11)$$

Therefore, S has a minimum at z_o.

To show that $A_o' = S(z_o) < 0$, suppose, to the contrary, that $S(z_o) \geq 0$. Then, since $S'(z) > 0$ for $z > z_o$, $S(z) > S(z_o) \geq 0$ for all $z > z_o$ and hence $\lim_{z\to\infty} S(z) > 0$. But $\lim_{z\to\infty} S(z) = \lim_{z\to\infty} \frac{\pi^2\lambda}{2IC}\Phi^2(z) - \pi\sigma \lim_{z\to\infty} u(z) = 0$ (see Appendix A). Thus, $A_o' = S(z_o) < 0$. This together with (11) proves part (a).

Now at $-z_o$,

$$S''(-z_o) = \pi\sigma\phi(-z_o)[\frac{\pi\lambda}{IC\sigma}(\frac{IC\sigma}{\pi\lambda} - z_o\Phi(-z_o)) - 1]$$

$$= \pi\sigma\phi(-z_o)[1 - \frac{\pi\lambda}{IC\sigma} z_o\Phi(-z_o) - 1]$$

$$= -\frac{\pi^2\lambda}{IC} z_o\phi(-z_o)\Phi(-z_o) < 0 . \qquad (12)$$

Therefore, S has a maximum at $-z_o$.

To show that $A_o = S(-z_o) \geq 0$, again suppose that $S(-z_o) < 0$. Since $\lim_{z\to\infty} S(z) = 0$, there exists some z' such that $S(z') > S(-z_o)$, for otherwise $S(z) \leq S(-z_o)$ for all z implies that $\lim_{z\to\infty} S(z) \leq S(-z_o) < 0$. But the existence of such z' contradicts the fact that S is maximized at $-z_o$. Thus, $A_o = S(-z_o) \geq 0$. This result coupled with (12) proves part (b).

Part (c) can now be easily proved by noting that for $z < -z_o < 0$,

$$\phi(z) + z\Phi(z) < \phi(z) < \phi(-z_o) = \frac{IC\sigma}{\pi\lambda}$$

hence $\frac{\pi\lambda}{IC\sigma}(\phi(z) + z\Phi(z)) < 1$

or $\frac{\pi\lambda}{IC\sigma}(\phi(z) + z\Phi(z)) - 1 < 0$

and therefore $S''(z) = \pi\sigma\phi(z)[\frac{\pi\lambda}{IC\sigma}(\phi(z) + z\Phi(z)) - 1] < 0$

Thus, S is strictly concave increasing on the interval $(-\infty,-z_o)$.

The proof of Lemma 3 is now complete.

Q.E.D.

REMARK: The minimum and maximum in Lemma 2 are clearly unique. Moreover, $\lim_{z \to -\infty} S(z) = -\infty$ (see Appendix A). A graph of S(z) corresponding to the example of Section II is given in Fig. 3-1.

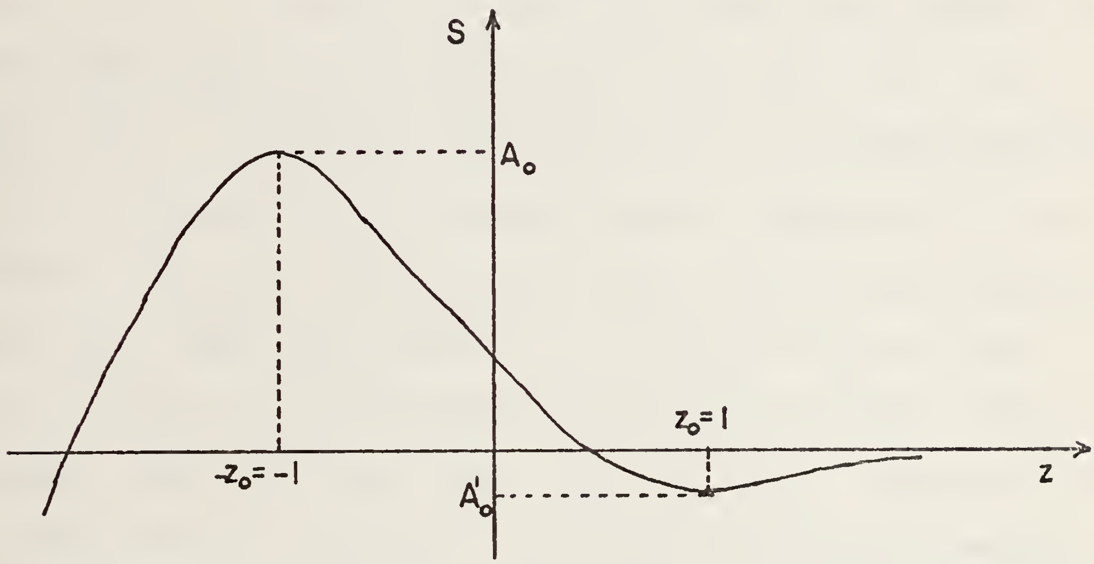

Fig. 3-1.

The following lemma shows the importance of A in the characterization of a solution.

LEMMA 4. Suppose $\frac{IC\sigma}{\pi\lambda} < \phi(0)$ and $A \geq 0$ is given. Then (Q^*, z^*) is a solution if and only if $A = S(z^*)$.

Proof. The necessary part has been proved by Lemma 1, part (a). The sufficient condition will now be proved. Suppose $A = S(z^*) = \frac{\pi^2\lambda}{2IC}\,\Phi^2(z^*) - \pi\sigma u(z^*)$. Let $Q^* = \frac{\pi\lambda\Phi(z^*)}{IC}$.

Then, it can be easily shown by substitution of A and Q* into equations (6) and (7) that (Q^*, z^*) is a solution.

Q.E.D.

REMARK: It should be observed immediately from Lemmas 4 and 3 that a necessary condition for a solution to exist is that $0 \leq A \leq A_o$. The following theorem summarizes all the results.

THEOREM. Given π, λ, I, C, σ all positive and $\frac{IC\sigma}{\pi\lambda} < \phi(0)$ and $A_o > 0$, then for $0 \leq A < A_o$ there exist exactly two solutions (Q_1^*, z_1^*) and (Q_2^*, z_2^*) such that $z_1^* > z_2^*$; (Q_1^*, z_1^*) is a minimum and (Q_2^*, z_2^*) is a saddle point for K.

Proof. Since $S(z)$ is strictly concave increasing in the interval $(-\infty, -z_o)$, and strictly decreasing in the interval $(-z_o, z_o)$ with $\lim_{z \to -\infty} S(z) = -\infty$, $S(-z_o) = A_o > 0$ and $S(z_o) = A_o' < 0$, the function $S(z)$ must have two roots z_1 and z_2 where $z_1 < -z_o$ and $-z_o < z_2 < z_o$. Thus, for any A such that $0 \leq A < A_o$, there are two distinct values z_1^* and z_2^* such that $z_1 < z_2^* < -z_o < z_1^* < z_2$ and $A = S(z_1^*) = S(z_2^*)$. By merely defining $Q_1^* = \frac{\pi\lambda\Phi(z_1^*)}{IC}$ and $Q_2^* = \frac{\pi\lambda\Phi(z_2^*)}{IC}$, it follows by Lemma 4 that (Q_1^*, z_1^*) and (Q_2^*, z_2^*) are solutions. But, $-z_o < z_1^* < z_o$ implies that $\phi(z_1^*) > \frac{IC\sigma}{\pi\lambda}$, and hence by Lemma 1, part (b), $V(z_1^*) > 0$. Thus, (Q_1^*, z_1^*) minimizes K. Similarly, $z_2^* < -z_o$ implies that $\phi(z_2^*) < \frac{IC\sigma}{\pi\lambda}$ and hence by the same lemma $V(z_2^*) < 0$. Thus, (Q_2^*, z_2^*) is a saddle point.

Q.E.D.

REMARK: For $A = A_o = S(-z_o)$, the only solution possible is determined by $z^* = -z_o$. But then, $\phi(z^*) = \frac{IC\sigma}{\pi\lambda}$ and thus $V(z^*) = 0$, so that no conclusion can be drawn.

IV. COMPUTATIONAL ASPECTS

Before attempting to solve a given problem involving a set of parameters, some simple tests should be performed to see whether or not there is any solution.

1) The first test is $\frac{IC\sigma}{\pi\lambda} < \phi(0)$.

2) If this test is passed, then find $-z_o$ and $A_o = S(-z_o)$.

Two cases could then occur.

Case (i): $A_o = 0$. The only solution then is $-z_o$ and $V(-z_o) = 0$ and it has been pointed out that no conclusion may then be drawn.

Case (ii): $A_o > 0$. If $0 \leq A < A_o$, then there exist exactly two solutions (Q_1^*, z_1^*) and (Q_2^*, z_2^*) with $z_2^* < -z_o < z_1^*$. As shown, (Q_1^*, z_1^*) is a minimum and (Q_2^*, z_2^*) is a saddle point. If $A = A_o$, then again the only solution is $-z_o$ and no conclusion can be drawn as in case (i).

Hadley and Whitin give an iterative scheme for solving for (Q^*, r^*) in Ref. 1. This scheme seems to converge always to the minimum value and never to the saddle point. But there is a simple explanation for this. The scheme begins by choosing initially a Wilson $Q_W = \sqrt{\frac{2\lambda A}{IC}}$ for the value of Q to be used in Eq. (9) (see Fig. 2-1) and $Q_W < Q_1^*$. The iteration then proceeds to choose a value of r corresponding to Q_W from Eq. (8). With that value of r a new value of Q is selected

from Eq. (9) and it will always be the case that $Q < Q_1^*$ in that selection. Proceeding in this manner, the pairs (Q,r) converge to the solution (Q_1^*,r_1^*) and cannot yield (Q_2^*,r_2^*). Thus, when a minimum exists, this computational scheme may be relied upon.

However, in any case of doubt, a simple test is furnished by checking whether or not $\phi(z^*) > \frac{IC\sigma}{\pi\lambda}$ for the solution.

It has been pointed out that the boundary case $A = A_o$ causes problems in that no conclusion may then be drawn. This presents no difficulty for practical uses, however. After all, an optimal solution may be given for any A arbitrarily close to A_o. In an application it would be a rare circumstance that produced a set of parameters in which A (typically measured in dollars and cents) would be exactly A_o.

V. CONCLUSION

While the results of this thesis only apply to the case of normal lead-time demand, they are of more than mere academic interest. In fact, that is a common assumption made by Hadley and Whitin along with users of the formulas and schemes given by them. And it is a happy coincidence that the algorithms given are indeed valid even though the assumptions under which they were derived were replete with errors.

Nevertheless, it is a recommendations of this thesis that further study into other types of lead-time distributions be made. One of the key features of the results derived here was the examination of the Hessian matrix with the observation that its positivity did not depend on Q. This will still be true with a density other than normal and so much of the same general technique ought to apply.

One of the features of this thesis has been to amplify and clarify completely the minimum solution. Conditions are given for existence,and formulas are given along with simple test criteria when existence is assured. A by-product of the results is the fact that negative safety levels do indeed occur as the unique optimal solution. In fact, a simple examination of a graph like S(z) will show immediately and simplify those values of the set-up cost A for which such solutions are valid. Since the solution in such cases is

unique, certainly negative safety levels should not be ignored or tampered with (truncation to μ for example) in an application. It should also be noted that the solutions are characterized independently of μ.

As a final recommendation, this thesis was restricted to the so-called backorders model where a customer must wait for delivery when the inventory position is out of stock. There are, however, other models such as the "lost sale case" discussed by Hadley and Whitin to apply in this situation. An analysis similar to that applied here is surely called for so that resultant solutions are validated and characterized. For the authors continually appeal to convexity for drawing conclusions in those cases.

APPENDIX A. VARIATION OF V(z)

As in Section II,

$$\Omega = \begin{bmatrix} \frac{2\lambda}{Q^3}(A + \pi\bar{\eta}(z)) & \frac{\pi\lambda\Phi(z)}{Q^2} \\ \frac{\rho\lambda\Phi(z)}{Q^2} & \frac{\pi\lambda\phi(z)}{\sigma Q} \end{bmatrix}$$

The diagonal term $\frac{\pi\lambda\phi(z)}{\sigma Q}$ is certainly positive.

$\bar{\eta}(z) = \sigma(\phi(z) - z\Phi(z)) = \sigma u(z)$ as defined in (4).

$$u'(z) = -z\phi(z) - (\Phi(z) - z\phi(z))$$

$$= -\Phi(z) < 0 \qquad \text{for all } z$$

$$u''(z) = \phi(z) > 0 \qquad \text{for all } z$$

Clearly, $\lim\limits_{z\to-\infty} u(z) = \lim\limits_{z\to-\infty} (\phi(z) - z\Phi(z)) = 0 + \infty = +\infty$

and

$$\lim_{z\to\infty} u(z) = \lim_{z\to\infty} (\phi(z) - z\Phi(z)) = - \lim_{z\to\infty} z\Phi(z)$$

$$= - \lim_{z\to\infty} \frac{z}{\frac{1}{\Phi(z)}} \quad .$$

Applying L'Hospital's Rule repeatedly,

$$\lim_{z\to\infty} u(z) = -\lim_{z\to\infty} \frac{1}{\frac{\phi(z)}{\Phi^2(z)}} = -\lim_{z\to\infty} \frac{\Phi^2(z)}{\phi(z)} = -\lim_{z\to\infty} \frac{-2\Phi(z)\phi(z)}{-z\phi(z)}$$

$$= -\lim_{z\to\infty} \frac{2\Phi(z)}{z} = 0 \ .$$

Thus, $u(z)$ is a strictly convex decreasing function approaching zero as $z \to +\infty$ and hence $u(z) > 0$ for all z. A sketch of $u(z)$ is given in Fig. A-1.

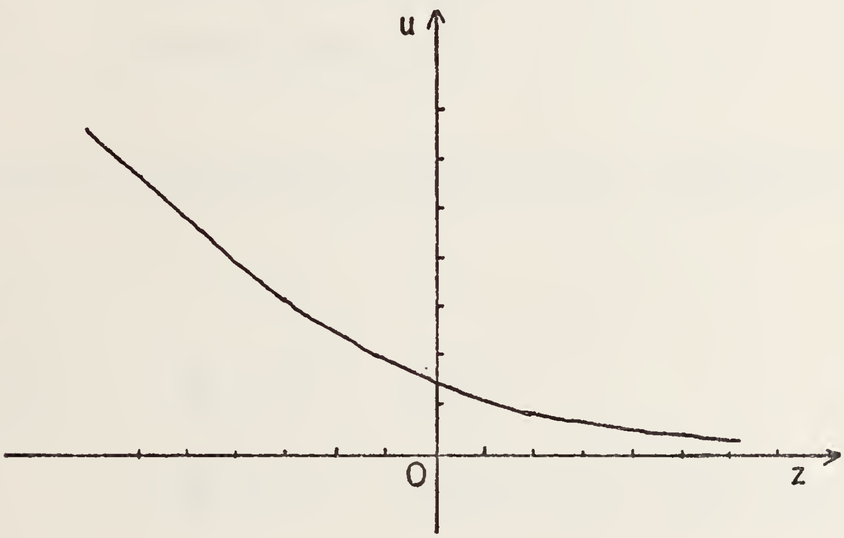

Fig. A-1

Since $u(z) > 0$ for all z, $\bar{\eta}(z) = \sigma u(z) > 0$ for all z and therefore the diagonal term $\frac{2\lambda}{Q^3}(A + \pi\bar{\eta}(z)) > 0$ for all z since $A \geq 0$.

$$V(z) = \frac{2A}{\sigma}\,\phi(z) + 2\pi\phi^2(z) - 2\pi z\phi(z)\Phi(z) - \pi\Phi^2(z)$$

Define $T(z) = 2\phi^2(z) - 2z\phi(z)\Phi(z) - \Phi^2(z)$, then

$$V(z) = \frac{2A}{\sigma}\,\phi(z) + \pi T(z)$$

$$\begin{aligned} T'(z) &= -4z\phi^2(z) + 2z^2\phi(z)\Phi(z) + 2z\phi^2(z) \\ &= 2z^2\phi(z)\Phi(z) - 2z\phi^2(z) \\ &= 2z\phi(z)(z\Phi(z) - \phi(z)) \\ &= -2z\phi(z)u(z) \end{aligned} \tag{A.1}$$

$$T''(z) = -2\phi(z)u(z) + 2z^2\phi(z)u(z) + 2z\phi(z)\Phi(z) \tag{A.2}$$

Thus,

$$\begin{aligned} V'(z) &= -\frac{2A}{\sigma}\,z\phi(z) + \pi T'(z) \\ &= -\frac{2A}{\sigma}\,z\phi(z) - 2\pi z\phi(z)u(z) \quad \text{from (A.1)} \\ &= -2z\phi(z)\left(\frac{A}{\sigma} + \pi u(z)\right) \end{aligned}$$

Since $u(z) > 0$ for all z

$$V'(z) \begin{cases} > 0 & \text{for} \quad z < 0 \\ = 0 & \text{for} \quad z = 0 \\ < 0 & \text{for} \quad z > 0 \end{cases}$$

$$V''(z) = -\frac{2A}{\sigma}\phi(z) + \frac{2A}{\sigma}z^2\phi(z) + \pi T''(z)$$

At $z = 0$,

$$V''(0) = -\frac{2A}{\sigma}\phi(0) + \pi T''(0)$$

$$= -\frac{2A}{\sigma}\phi(0) - 2\pi\phi(0)u(0) \qquad \text{from (A.2)}$$

Thus $V''(0) < 0$ and hence $V(z)$ is maximized at $z = 0$ and

$$V(0) = \frac{2A}{\sigma}\phi(0) + \pi T(0) = \frac{2A}{\sigma}(.3989) + \pi(.0683) > 0 .$$

$$\lim_{z\to+\infty} V(z) = 0 + \pi \lim_{z\to+\infty} T(z) = 0$$

$$\lim_{z\to-\infty} V(z) = 0 + \pi \lim_{z\to-\infty} T(z) = \pi(-1) = -\pi .$$

A sketch of $V(z)$ is given in Fig. A-2.

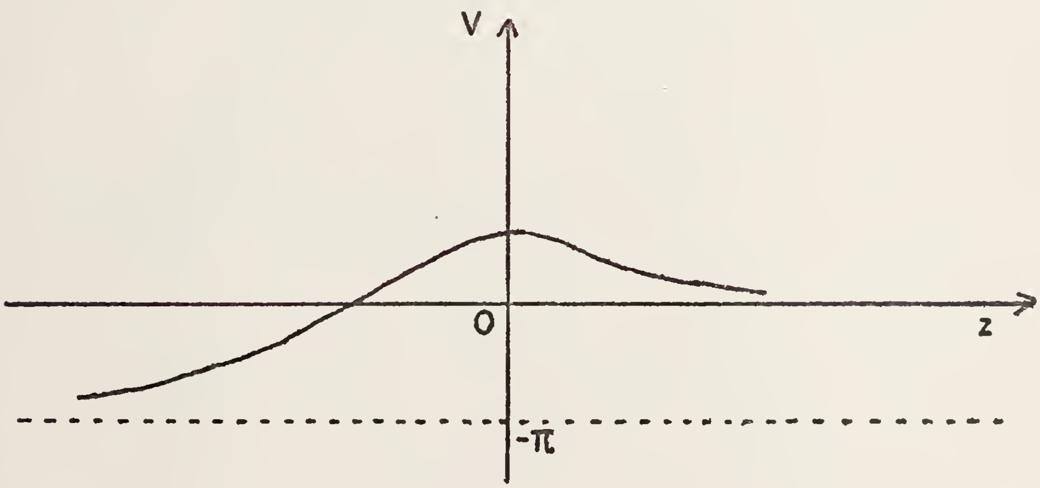

Fig. A-2.

Since $V(z)$ attains a maximum with positive value at $z = 0$ and is decreasing for $z > 0$ and $\lim_{z\to+\infty} V(z) = 0$, $V(z) > 0$ for all $z \geq 0$.

BIBLIOGRAPHY

1. Hadley, G. and Whitin, T.M., Analysis of Inventory Systems, Prentice Hall, 1963.

2. Veinott, A.F., "Review of Hadley and Whitin," Journal of the American Statistical Association, v. 59, March 1964, pp. 283-285.

3. Brooks, R.S. and Lu, J.Y., "On the Convexity of the Backorder Function for an E.O.Q. Policy," Management Science, v. 15, no. 7, March 1969, pp. 453-454.

4. Roberts, A.N. and Varberg, D.E., Convex Functions, Academic Press, 1973.

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